On linear forms containing the Euler constant*

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1 Introduction

We study the arithmetical nature of two numbers

$$\gamma := -\int_0^\infty \ln x\, e^{-x} dx \qquad \text{and} \qquad \delta := \int_0^\infty \ln(x+1)\, e^{-x} dx\,.$$

The first number is the famous Euler (or Euler-Mascheroni) constant

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) \approx 0.577\ldots\,,$$

The second number is called Euler-Gompertz constant (see [1]). A relation

$$\delta := \int_0^\infty \frac{e^{-x}dx}{x+1} \approx 0.596\ldots$$

to the Laguerre polynomials gives a sequence of rational approximants for δ

$$\frac{\tilde{p}_n}{\tilde{q}_n} \to \delta\,,\ n\to\infty\,, \tag{1}$$

generated by the recurrence relations

$$\tilde{q}_{n+1} \;=\; 2\,(n+1)\,\tilde{q}_n \;-\; n^2\,\tilde{q}_{n-1}\,,$$

with initial condition

$$\begin{aligned} \tilde{p}_0 &= 0\,, \quad \tilde{p}_1 = 1\,, \\ \tilde{q}_0 &= 1\,, \quad \tilde{q}_1 = 2\,. \end{aligned}$$

The Perron asymptotics for the Laguerre polynomials

$$\tilde{q}_n \;=\; n!\,\frac{e^{2\sqrt{n}}}{\sqrt[4]{n}}\left(\frac{1}{2\sqrt{\pi e}} + O(n^{-1/2})\right)\,,$$

$$\tilde{p}_n - \delta\tilde{q}_n \;=\; O\left(n!\,\frac{e^{-2\sqrt{n}}}{\sqrt[4]{n}}\right)\,,$$

confirm (1). However, the rational approximants (1) do not imply that δ is an irrational number. As far as we know, irrationality of δ is still an open problem, as well as the famous open problem about irrationality of the Euler constant γ.

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2 Result

Theorem 1. *Given sequences of numbers*

$$u := \{u_n\}\,, \quad v := \{v_n\}\,, \quad w := \{w_n\}\,,$$

generated by the recurrence relations

$$\begin{aligned}(16n+1)(16n-15)\,u_{n+1} \;=\; & (16n-15)(256n^3+528n^2+352n+73)\,u_n \\ & -\;(16n+17)(128n^3+40n^2-82n-45)\,u_{n-1} \\ & +\;n^2(16n+17)(16n+1)\,u_{n-2}\,,\end{aligned} \tag{2}$$

with initial conditions

$$\begin{aligned} u_0 &= -2\,,\ u_1 = 7\,,\ u_2 = 558\,,\\ v_0 &= -1\,,\ v_1 = -22\,,\ v_2 = -1518\,,\\ w_0 &= 0\,,\ w_1 = -17\,,\ w_2 = -1209\,,\end{aligned}$$

Then

1) $u_n,\, v_n\,, w_n \in \mathbb{Z}$;

2) difference equation (2) *has solutions with three different asymptotics as* $n \to \infty$

$$\begin{aligned} u_n\,,\ v_n\,,\ w_n \;&=\; O\left(\frac{(2n)!\,4^n}{n^{3/2}}\right)\,,\\ [w_n+(e\gamma+\delta)u_n]\,,\ [v_n+eu_n] \;&=\; O\left(\frac{e^{\sqrt{2n}}\,n^{5/4}}{4^n}\right)\,,\\ l_n := u_n\delta-\gamma v_n+w_n \;&=\; \frac{n^{5/4}}{e^{\sqrt{2n}}\,4^n}\left(\frac{2\sqrt[4]{2}}{e^{3/8}}+O(n^{-1/2})\right)\,.\end{aligned} \tag{3}$$

Remark 1. The asymptotics of l_n in (3) give a quantitative characterization of the fact that one of two constants γ or δ is an irrational number. The validity of this fact was known before, it follows from the A.B. Shidlovski result [2] on algebraic independence of the values of E-functions (see also K. Mahler [3]). Indeed (see, for example, the last statement of [3]), numbers

$$1-\frac{1}{e} \quad \text{and} \quad -\left(\gamma-\frac{\delta}{e}\right)$$

are algebraically independent. It implies that numbers γ and δ can not be rational numbers simultaneously.

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3 A rational approximation of Euler constant

The numbers generated by (2) are intimately related with rational approximants of the Euler constant (studied in [4]). These approximants are produced by the functional Hermite-Pade rational approximants for a system of four functions $\{\hat{\mu}_1,\, \hat{\mu}_2,\, \hat{s}_1,\, \hat{s}_2\}$:

$$\hat{\mu}_k(z) \;:=\; \int_0^1 \frac{w_k(z)dx}{z-x}\,, \qquad \hat{s}_k(z) \;:=\; \int_1^\infty \frac{w_k(z)dx}{z-x}\,, \tag{4}$$

where

$$w_k(x) \;:=\; x^{\alpha_k}\,(1-x)^\alpha\, e^{-\beta x}\,, \qquad k=1,2\,. \tag{5}$$

For the denominators Q_n of the Hermite-Pade rational approximants of this system (4)-(5) the generalized Rodrigues formula was known (see the example of subsection 2.2 in [5]; for similar systems, see also [6] and [7])

$$Q_n(z) \;=\; \frac{1}{(n!)^2}\, w_2^{-1}\frac{d^n}{dz^n}\left[w_2 z^n w_1^{-1}\frac{d^n}{dz^n}\left[w_1 z^n(1-z)^{2n}\right]\right]\,. \tag{6}$$

Then the above mentioned rational approximants $\dfrac{p_n}{q_n}$ of the Euler constant γ are defined as

$$p_n \,-\, \gamma\, q_n \;:=\; \int_0^\infty Q_n(x)\ln(x)\, e^{-x}\, dx \;=:\; f_n, \tag{7}$$

where Q_n is taken from (6) – (5) with parameters $\alpha_1=\alpha_2=0\;\;\alpha=-\beta=1$.

In [8]–[9] recurrence relations (of six, seven and eight terms) for polynomials (6) were studied, which in [10] eventually brought a four-term recurrence relation for sequences of numbers $p:=\{p_n\}$ and $q:=\{q_n\}$

$$\begin{aligned}(16n-15)\,q_{n+1} \;&=\; (128n^3+40n^2-82n-45)\,q_n \\ &\quad-\; n^2(256n^3-240n^2+64n-7)\,q_{n-1} \;+\; n^2(n-1)^2(16n+1)\,q_{n-2}\,,\end{aligned} \tag{8}$$

with initial conditions

$$\begin{aligned}p_0=0\,,\; p_1=2\,,\; p_2=31\,,\\ q_0=1\,,\; q_1=3\,,\; q_2=50\,.\end{aligned}$$

We also highlight a solution $r:=\{r_n\}$ of difference equation (8) with initial conditions

$$r_0=0\,,\; r_1=1\,,\; r_2=24\,.$$

The fact that numbers $q_n,\, p_n,\, r_n\,\in\mathbb{Z}$ are integers for $n\in\mathbb{N}$ was proven in [13]. Finally in [11] and [12], the following asymptotics for $\{q_n,\, p_n,\, r_n\}$ and the linear forms with these coefficients were obtained

$$\begin{aligned} q_n\,,\; p_n\,,\; r_n \quad &=\; O\,(\frac{(2n)!\,e^{\sqrt{2n}}}{\sqrt[4]{n}})\,, \\ f_n := p_n-\gamma q_n \quad &=\; (2n)!\,\frac{e^{-\sqrt{2n}}}{\sqrt[4]{n}}\left(\frac{2\sqrt{\pi}}{(4e)^{3/8}}+O(n^{-1/2})\right)\,, \\ g_n := ep_n-(e\gamma+\delta)q_n+r_n \quad &=\; \frac{1}{16^n}\left(\frac{1}{8}+O(n^{-1})\right)\,. \end{aligned} \tag{9}$$

4 Proof of Theorem 1

A) We define

$$u_n := \frac{\Delta_n^{(qp)}}{(n!)^2}\,, \qquad v_n := \frac{\Delta_n^{(qr)}}{(n!)^2}\,, \qquad w_n := \frac{\Delta_n^{(pr)}}{(n!)^2}\,, \tag{10}$$

where we use a notation

$$\Delta_n^{(ab)} := \det\begin{pmatrix} a_{n+1}\,, & a_n \\ b_{n+1}\,, & b_n \end{pmatrix}\,, \qquad a := \{a_n\}\,, \quad b := \{b_n\}\,. \tag{11}$$

B) Substituting the recurrence relations

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = A_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} + B_n \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} + C_n \begin{pmatrix} a_{n-2} \\ b_{n-2} \end{pmatrix}$$

into determinant (11), we get

$$\begin{vmatrix} a_{n+1}\,, & a_n \\ b_{n+1}\,, & b_n \end{vmatrix} = -B_n \begin{vmatrix} a_n\,, & a_{n-1} \\ b_n\,, & b_{n-1} \end{vmatrix} - C_n A_{n-1} \begin{vmatrix} a_{n-1}\,, & a_{n-2} \\ b_{n-1}\,, & b_{n-2} \end{vmatrix} + C_n C_{n-1} \begin{vmatrix} a_{n-2}\,, & a_{n-3} \\ b_{n-2}\,, & b_{n-3} \end{vmatrix}\,,$$

that from (8) leads to (2).

C) The fact that $\dfrac{\Delta_n^{(qp)}}{(n!)^2} \in \mathbb{Z}$ is proven in [14]. We get that v_n, $w_n \in \mathbb{Z}$ analogously.

D) The asymptotics of

$$\Delta_n^{(qp)} = \begin{vmatrix} q_{n+1} & , \; q_n \\ p_{n+1} - \gamma q_{n+1}\,, & p_n - \gamma q_n \end{vmatrix} = O\left(\frac{(2n)!^2}{n^{3/2}}\right)$$

was computed in [14]. The same way, from (9) we deduce

$$\Delta_n^{(qr)} = \begin{vmatrix} q_{n+1} & , \; q_n \\ r_{n+1} - \gamma q_{n+1}\,, & r_n - \gamma q_n \end{vmatrix} = O\left(\frac{(2n)!^2}{n^{3/2}}\right)\,.$$

Noticing that

$$\Delta_n^{(pr)} \frac{1}{e} - \Delta_n^{(pq)} \left(\gamma + \frac{\delta}{e}\right) = \Delta_n^{(pg)}\,,$$

we get from (9) asymptotics for $[w_n + (e\gamma + \delta)\, u_n]$ and w_n. Analogously, the identity

$$\Delta_n^{(qp)} + \Delta_n^{(qr)} \frac{1}{e} = \Delta_n^{(qg)}$$

brings the asymptotics for $(v_n + e\, u_n)$. Finally,

$$-\Delta_n^{(pq)} \frac{\delta}{e} + \Delta_n^{(pr)} \frac{1}{e} - \Delta_n^{(qr)} \frac{\gamma}{e} = \Delta_n^{(fg)}$$

from (9) brings the asymptotics for l_n in (3).

The Theorem is proved.

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Abstract

We present linear forms with integer coefficients containing the Euler-Mascheroni and Euler-Gompertz constants. The forms are defined by four-terms recurrence relations. Asymptotics of the forms and their coefficients are obtained.